

# Group Extensions of the Co-type of a Crossed Module and Strict Categorical Groups

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## Abstract

*Prolongations* of a group extension can be studied in a more general situation that we call group extensions of the co-type of a crossed module. Cohomology classification of such extensions is obtained by applying the obstruction theory of monoidal functors.

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## 1 Introduction

A description of group extensions by means of factor sets leads to a close relationship between the extension problem of a type of algebras and the corresponding cohomology theory. This allows to study extension problems using cohomology as an effective method [6].

Let  $A$  and  $\Pi$  be two groups,  $A$  abelian. An extension of  $A$  by  $\Pi$  is a short exact sequence

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} \Pi \rightarrow 1. \quad (1)$$

A classical theorem in homological algebra asserts that the group of isomorphism classes of extensions of  $A$  by  $\Pi$  with a fixed operator  $\varphi : \Pi \rightarrow \text{Aut} A$  is isomorphic to the second cohomology group  $H^2_\varphi(\Pi, A)$ , [9]. After that, the group  $H^2$  was applied to the problem of classifying all group extensions in the different situations. This theorem has been made more precise by establishing a categorical equivalence between the category of extensions and a certain category whose objects are 2-cocycles [8]. With the notion of a *categorical group* (or a *Gr-category* [14]), many aspects of group extension problem are raised to a categorical level which help to obtain applications in algebra (see [4], [15]). This article belongs to this type.

The article is derived from the following classical problem. For a group extension (1) and a group homomorphism  $\eta : \Pi' \rightarrow \Pi$ , it follows from the existence of the pull-back of the pair  $(\eta, p)$  that there is an extension  $E\eta$  making the following diagram commute

$$\begin{array}{ccccccc} E\eta : & 0 & \longrightarrow & A & \xrightarrow{j'} & B' & \xrightarrow{p'} \Pi' \longrightarrow 1 \\ & & & \parallel & & \downarrow \beta & \downarrow \eta \\ E : & 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} \Pi \longrightarrow 1 \end{array}$$

(see [9] - Chapter III, [5] - Chapter IV).

The problem is that with a given extension  $E'$  and a homomorphism  $\eta : \Pi' \rightarrow \Pi$ , let us find all extensions  $E$  of  $A$  by  $\Pi$  such that  $E' = E\eta$ . Then, the extension  $E$  is said to be a  $\eta$ -*prolongation* of  $E'$ . A brief and general description of this problem was introduced in [16] (Proposition 5.1.1). In [13] we show the better descriptions in the case of the central extensions and  $\eta$  is an injection. Each prolongation induces a model which is “dual” to a group extension of the type of a crossed module (see Section 6). This leads to the notion of group *extension of co-type* of a crossed module studied in this paper.

The plan of this paper is, briefly, as follows. In Section 2 we recall reduced categorical groups, monoidal functors of type  $(\varphi, f)$ . In Section 3 we show the relation between the category of crossed modules and the category of strict Gr-categories, which is a useful tool in the next proofs. Next, we introduce the notion of a  $\zeta$ -extension of the co-type of a crossed module in Section 4, and we construct the obstruction theory of a  $\zeta$ -extension (Theorem 6). In Section 5 we present Schreier theory for  $\zeta$ -extensions of the co-type of a crossed module (Theorem 9). The last section is devoted to applying the results of previous sections to the problem of prolongations of a group extension in [13].

## 2 Preliminaries

For later use, we recall here some basic facts and results about categorical groups (see [12], [14]).

A *categorical group* is a monoidal category  $(\mathbb{G}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$  in which every object is invertible and the underlying category is a groupoid. If  $(F, \tilde{F}, F_*)$  is a monoidal functor between categorical groups, the isomorphism  $F_* : I' \rightarrow FI$  can be deduced from  $F$  and  $\tilde{F}$ . Thus, we will refer to  $(F, \tilde{F})$  as a monoidal functor.

Two monoidal functors  $(F, \tilde{F})$  and  $(F', \tilde{F}')$  from  $\mathbb{G}$  to  $\mathbb{G}'$  are *homotopic* if there is a *natural monoidal equivalence* (or a *homotopy*)  $\alpha : (F, \tilde{F}, F_*) \rightarrow (F', \tilde{F}', F'_*)$ , which is a natural equivalence such that  $F'_* = \alpha_I \circ F_*$ .

Each categorical group  $\mathbb{G}$  determines three invariants, as follows:

1. The set  $\pi_0\mathbb{G}$  of isomorphism classes of the objects in  $\mathbb{G}$  is a group where the operation is induced by the tensor product in  $\mathbb{G}$ .
2. The set  $\pi_1\mathbb{G}$  of automorphisms of the unit object  $I$  is a  $\pi_0\mathbb{G}$ -module.
3. An element  $[k] \in H^3(\pi_0\mathbb{G}, \pi_1\mathbb{G})$  is induced by the associativity constraint of  $\mathbb{G}$ .

Based on the data: a group  $\Pi$ , a  $\Pi$ -module  $A$  and  $k \in Z^3(\Pi, A)$ , we construct a categorical group, denoted by  $\text{Red}(\Pi, A, k)$  whose objects are elements  $x \in \Pi$  and the morphisms are automorphisms  $(x, a) : x \rightarrow x$ , where  $x \in \Pi, a \in A$ . The composition of two morphisms is induced by the

addition in  $A$

$$(x, a) \circ (x, b) = (x, a + b).$$

The tensor products are given by

$$\begin{aligned} x \otimes y &= x.y, \quad x, y \in \Pi, \\ (x, a) \otimes (y, b) &= (xy, a + xb), \quad a, b \in A. \end{aligned}$$

The unit constraints of the categorical group  $\text{Red}(\Pi, A, k)$  are strict, and its associativity constraint is  $\mathbf{a}_{x,y,z} = (xyz, k(x, y, z))$ .

In the case where  $\Pi, A, [k]$  are three invariants of a categorical group  $\mathbb{G}$  then  $\text{Red}(\Pi, A, k)$  is monoidally equivalent to  $\mathbb{G}$  and it is called a *reduction* of  $\mathbb{G}$ , hence denoted by  $\mathbb{G}(k)$ .

A functor  $F : \text{Red}(\Pi, A, k) \rightarrow \text{Red}(\Pi', A', k')$  is of *type*  $(\varphi, f)$  if

$$F(x) = \varphi(x), \quad F(x, a) = (\varphi(x), f(a)),$$

where  $\varphi : \Pi \rightarrow \Pi', f : A \rightarrow A'$  are group homomorphisms satisfying  $f(xa) = \varphi(x)f(a)$ , for  $x \in \Pi, a \in A$ . Note that if  $\Pi'$ -module  $A'$  is considered as a  $\Pi$ -module under the action  $xa' = \varphi(x).a'$ , then  $f : A \rightarrow A'$  is a homomorphism of  $\Pi$ -modules. In this case, we call  $(\varphi, f)$  a *pair of homomorphisms* and call

$$\xi = \varphi^* k' - f_* k \in Z^3(\Pi, A') \quad (2)$$

an *obstruction* of the functor  $F$ , where  $\varphi^*, f_*$  are canonical homomorphisms

$$Z^3(\Pi, A) \xrightarrow{f_*} Z^3(\Pi, A') \xleftarrow{\varphi^*} Z^3(\Pi', A').$$

The results on monoidal functors of type  $(\varphi, f)$  stated in [12] are summarized in the following proposition.

**Proposition 1.** *Let  $\mathbb{G}$  and  $\mathbb{G}'$  be two categorical groups,  $\mathbb{G}(k)$  and  $\mathbb{G}'(k')$  be their reductions, respectively.*

- i) *Every monoidal functor  $(F, \tilde{F}) : \mathbb{G} \rightarrow \mathbb{G}'$  induces one  $\mathbb{G}(k) \rightarrow \mathbb{G}'(k')$  of type  $(\varphi, f)$ .*
- ii) *Every monoidal functor  $\mathbb{G}(k) \rightarrow \mathbb{G}'(k')$  is a functor of type  $(\varphi, f)$ .*
- iii) *A functor  $F : \mathbb{G}(k) \rightarrow \mathbb{G}'(k')$  of type  $(\varphi, f)$  is realizable, that is, it induces a monoidal functor, if and only if its obstruction  $[\xi]$  vanishes in  $H_\Gamma^3(\Pi, A')$ . Then, there is a bijection*

$$\text{Hom}_{(\varphi, f)}[\mathbb{G}(k), \mathbb{G}'(k')] \leftrightarrow H_\Gamma^2(\Pi, A'),$$

where  $\text{Hom}_{(\varphi, f)}[\mathbb{G}(k), \mathbb{G}'(k')]$  is the set of all homotopy classes of monoidal functors of type  $(\varphi, f)$  from  $\mathbb{G}(k)$  to  $\mathbb{G}'(k')$ .

### 3 Categorical groups associated to a crossed module

A categorical group is *strict*, according to Joyal and Street [7], if all of its constraints are strict and every object has a strict inverse ( $x \otimes y = 1 = y \otimes x$ ). Brown and Spencer [3] called it a  $\mathcal{G}$ -groupoid. The authors of [3] showed that there is a categorical equivalence between the category of crossed modules and that of  $\mathcal{G}$ -groupoids, and hence crossed modules can be studied by means of category theory. The Brown-Spencer equivalence has recently developed for the category of (braided) crossed bimodules (see [10], Theorems 4.3, 4.4).

**Definition.** A *crossed module* is a quadruple  $(B, D, d, \theta)$  where  $d : B \rightarrow D$ ,  $\theta : D \rightarrow \text{Aut} B$  are group homomorphisms such that the following relations hold

$$\begin{aligned} C_1. \quad & \theta d = \mu, \\ C_2. \quad & d(\theta_x(b)) = \mu_x(d(b)), \quad x \in D, b \in B, \end{aligned}$$

where  $\mu_x$  is an inner automorphism given by conjugation of  $x$ .

**Definition.** A *homomorphism*  $(f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')$  of crossed modules consists of group homomorphisms  $f_1 : B \rightarrow B'$ ,  $f_0 : D \rightarrow D'$  satisfying

$$\begin{aligned} H_1. \quad & f_0 d = d' f_1, \\ H_2. \quad & f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b), \end{aligned}$$

for all  $x \in D, b \in B$ .

In the present paper, the crossed module  $(B, D, d, \theta)$  is sometimes denoted by  $B \xrightarrow{d} D$ . For convenience, we denote by the addition for the operation in  $B$  and by the multiplication for that in  $D$ .

The following properties follow from the definition of a crossed module.

**Proposition 2.** Let  $(B, D, d, \theta)$  be a crossed module.

- i)  $\text{Ker} d \subset Z(B)$ .
- ii)  $\text{Im} d$  is a normal subgroup in  $D$ .
- iii) The homomorphism  $\theta$  induces a homomorphism  $\varphi : D \rightarrow \text{Aut}(\text{Ker} d)$  given by

$$\varphi_x = \theta_x|_{\text{Ker} d}.$$

- iv)  $\text{Ker} d$  is a left  $\text{Coker} d$ -module with the action

$$sa = \varphi_x(a), \quad a \in \text{Ker} d, \quad x \in s \in \text{Coker} d.$$

As mentioned above, a categorical group can be seen as a crossed module [3], [7]. To help motivate the reader, we present this fact in detail.

For each crossed module  $(B, D, d, \theta)$ , one can construct a strict categorical group  $\mathbb{G}_{B \rightarrow D} = \mathbb{G}$ , called the categorical group *associated to* the crossed module  $B \rightarrow D$ , as follows.

$$\text{Ob} \mathbb{G} = D, \quad \text{Hom}(x, y) = \{b \in B \mid x = d(b)y\},$$

where  $x, y$  are objects of  $\mathbb{G}$ . The composition of two morphisms is given by

$$(x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).$$

The tensor functor is given by  $x \otimes y = xy$  and

$$(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (xx' \xrightarrow{b+\theta_y b'} yy'). \quad (3)$$

Conversely, for a strict categorical group  $(\mathbb{G}, \otimes)$ , we define a crossed module  $C_{\mathbb{G}} = (B, D, d, \theta)$  as follows. Set

$$D = \text{Ob}\mathbb{G}, \quad B = \{x \xrightarrow{b} 1 | x \in D\}.$$

The operations on  $D$  and on  $B$  are given by

$$xy = x \otimes y, \quad b + c = b \otimes c,$$

respectively. Then, the set  $D$  becomes a group in which the unit is 1, the inverse of  $x$  is  $x^{-1}$  ( $x \otimes x^{-1} = 1$ ). The set  $B$  is a group in which the unit is the morphism  $(1 \xrightarrow{id_1} 1)$  and the inverse of  $(x \xrightarrow{b} 1)$  is the morphism  $(x^{-1} \xrightarrow{\bar{b}} 1)(b \otimes \bar{b} = id_1)$ .

The homomorphisms  $d : B \rightarrow D$  and  $\theta : D \rightarrow \text{Aut } B$  are respectively given by

$$d(x \xrightarrow{b} 1) = x, \\ \theta_y(x \xrightarrow{b} 1) = (yxy^{-1} \xrightarrow{id_y + b + id_{y^{-1}}} 1).$$

The following result shows the relationship between homomorphisms of crossed modules and monoidal functors of associated categorical groups.

**Proposition 3** ([11]). *Let  $(f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')$  be a homomorphism of crossed module.*

i) *There is a functor  $F : \mathbb{G}_{B \rightarrow D} \rightarrow \mathbb{G}_{B' \rightarrow D'}$  given by*

$$F(x) = f_0(x), \quad F(b) = f_1(b),$$

where  $x \in \text{Ob}\mathbb{G}$ ,  $b \in \text{Mor}\mathbb{G}$ ,

ii) *Natural isomorphisms  $\tilde{F}_{x,y}$  together with  $F$  is a monoidal functor if and only if  $\tilde{F}_{x,y} = \varphi(\bar{x}, \bar{y})$ , where  $\varphi \in Z^2(\text{Coker } d, \text{Ker } d')$ .*

*Note.* In the category of  $\mathcal{G}$ -groupoids in [3], the morphisms  $(F, \tilde{F})$  satisfy  $\tilde{F} = id$ .

## 4 Group extensions of the co-type of a crossed module

In this section we introduce a concept which is “dual” to the concept of group extension of type  $B \xrightarrow{d} D$  in [1, 2]. As will be showed later, it is also regarded as a generalization of the prolongation problem of group extensions [13].

**Definition.** Let  $d : B \rightarrow D$  be a crossed module. A group *extension* of  $A$  of *co-type*  $B \xrightarrow{d} D$  is a diagram of group homomorphisms

$$\begin{array}{ccccccc} & & B & \xrightarrow{d} & D & & \\ & & \downarrow \beta & & \parallel & & \\ 0 & \longrightarrow & A & \xrightarrow{j} & E & \xrightarrow{p} & D \longrightarrow 1, \end{array}$$

where the bottom row is exact,  $j(A) \subset Z(E)$ , the pair  $(\beta, id_D)$  is a morphism of crossed modules.

Since the bottom row is exact and since  $p \circ \beta \circ i = d \circ i = 0$ , where  $i : \text{Ker } d \rightarrow B$  is an inclusion, there exists a unique homomorphism  $\zeta : \text{Ker } d \rightarrow A$  such that the left hand side square commutes

$$\begin{array}{ccccccc} & & \text{Ker } d & \xrightarrow{i} & B & \xrightarrow{d} & D \\ & & \downarrow \zeta & & \downarrow \beta & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{j} & E & \xrightarrow{p} & D \longrightarrow 1. \end{array} \quad (4)$$

This homomorphism is defined by

$$j(\zeta c) = \beta(ic), \quad c \in \text{Ker } d. \quad (5)$$

Moreover,  $\zeta$  depends only on the equivalence class of the extension  $E$ .

*Note on terminologies.* Since the homomorphism  $\theta'$  of the crossed module  $E \xrightarrow{p} D$  is the conjugation and since  $j(A) \subset Z(E)$ ,  $\theta'_x$  acts on  $A$  as an identity. Thus, the group  $A$  can be seen as a  $D$ -module with the trivial action. Then,

$$\zeta(sc) = \zeta(c), \quad s \in \text{Coker } d, c \in \text{Ker } d. \quad (6)$$

Indeed, By Proposition 2,  $\theta_x(c) \in \text{Ker } d$ , so one has

$$\begin{aligned} j\zeta(\theta_x(c)) &\stackrel{(5)}{=} \beta i(\theta_x c) = \beta(\theta_x c) \\ &\stackrel{(H_2)}{=} \theta'_x(\beta c) = \theta'_x(j\zeta(c)) = j\zeta(c). \end{aligned}$$

Since  $j$  is injective, we obtain (6). Thus, it defines a trivial  $\text{Coker } d$ -module structure on  $\text{Im } \zeta$ .

The homomorphism  $\zeta : \text{Ker } d \rightarrow A$  satisfying the condition (6) is called an *abstract  $\zeta$ -kernel* of the crossed module  $B \xrightarrow{d} D$ . An extension of  $A$  of co-type  $B \xrightarrow{d} D$  inducing  $\zeta : \text{Ker } d \rightarrow A$  is said to be an *extension of the abstract  $\zeta$ -kernel*, or a  $\zeta$ -*extension* of co-type  $B \xrightarrow{d} D$ .

• *The obstruction theory: the case  $\zeta$  is surjective*

From now on, assume that  $\zeta : \text{Ker } d \rightarrow A$  is an onto homomorphism. We use the obstruction theory of monoidal functors to deal with the existence of  $\zeta$ -extensions.

Let  $\mathbb{G} = \mathbb{G}_{B \rightarrow D}$  be the categorical group associated to crossed module  $B \rightarrow D$ . Since  $\pi_0 \mathbb{G} = \text{Coker } d$  and  $\pi_1 \mathbb{G} = \text{Ker } d$ , the reduced categorical group  $\mathbb{G}(k)$  is of form

$$\mathbb{G}(k) = \text{Red}(\text{Coker } d, \text{Ker } d, k), \quad [k] \in H^3(\text{Coker } d, \text{Ker } d),$$

where the associativity constraint  $k$  is defined as follows. Choose a set of representatives  $\{x_s \mid s \in \text{Coker } d\}$  in  $D$ . For each  $x \in s$  choose an element  $b_x \in B$  satisfying  $x_s = d(b_x)x$ ,  $b_{x_s} = 0$ . According to [14], the family  $(x_s, b_x)$  is called a *stick*. It defines a monoidal functor  $(H, \tilde{H}) : \mathbb{G}(k) \rightarrow \mathbb{G}$  by

$$H(s) = x_s, \quad H(s, a) = a, \quad \tilde{H}_{r,s} = -b_{x_r x_s}.$$

Then,  $k$  is determined by the following commutative diagram

$$\begin{array}{ccccc} x_s(x_r x_t) & \xrightarrow{x_s \otimes \tilde{H}_{r,t}} & x_s x_{rt} & \xrightarrow{\tilde{H}_{s,rt}} & x_{srt} \\ \parallel & & & & \downarrow k(s,r,t) \\ (x_s x_r) x_t & \xrightarrow{\tilde{H}_{s,r} \otimes x_t} & x_{sr} x_t & \xrightarrow{\tilde{H}_{sr,t}} & x_{rst}. \end{array} \quad (7)$$

By the relation (3), this diagram implies

$$\theta_{x_s}(\tilde{H}_{r,t}) + \tilde{H}_{s,rt} + k(s, r, t) = \tilde{H}_{s,r} + \tilde{H}_{sr,t}.$$

We write  $k = \delta(\tilde{H})$  even though the function  $\tilde{H}$  takes values in  $B$ . The cohomology class

$$\text{Obs}(\zeta) = [\zeta_* k] \in H^3(\text{Coker } d, A)$$

is called the *obstruction* of the abstract  $\zeta$ -kernel.

The onto homomorphism  $\zeta : \text{Ker } d \rightarrow A$  induces a quotient category  $\mathbb{G}/\text{Ker } \zeta$  with the same objects of  $\mathbb{G}(= D)$ , but morphisms are homotopy classes of morphisms in  $\mathbb{G}$ , i.e., elements of the group  $\overline{B} = B/\text{Ker } \zeta$ . The category  $\mathbb{G}/\text{Ker } \zeta$  is just the categorical group associated to the crossed module  $(\overline{B}, \overline{d}, \overline{\theta}, D)$  induced by the crossed module  $(B, d, \theta, D)$ .

**Lemma 4.** *If the obstruction  $\text{Obs}(\zeta)$  vanishes in  $H^3(\text{Coker } d, A)$ , there exists a monoidal functor  $\text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$ .*

*Proof.* If  $\text{Obs}(\zeta)$  vanishes in  $H^3(\text{Coker } d, A)$ , then  $\zeta_* k = \delta g$ , where  $g : (\text{Coker } d)^2 \rightarrow A$ . Consider a functor

$$F : \text{Red}(D, A, 0) \rightarrow \text{Red}(\text{Coker } d, A, \delta g),$$

for  $F = (q, id)$ , where  $q$  is the natural projection. The obstruction of  $F$  is

$$q^*(\delta g) = \delta(q^* g).$$

Thus,  $F$  together with  $\tilde{F} = q^* g$  is a monoidal functor. It follows the existence of a monoidal functor from  $\text{Red}(D, A, 0)$  to  $\mathbb{G}/\text{Ker } \zeta$ .  $\square$

**Lemma 5.** *Each monoidal functor  $\text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$  defines a  $\zeta$ -extension of co-type  $B \xrightarrow{d} D$ .*

*Proof.* Construction of the crossed product from a monoidal functor  $(\Gamma, \tilde{\Gamma}) : \text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$ .

The morphism  $\tilde{\Gamma}$  defines an associated function  $g : D^2 \rightarrow A$  by  $\tilde{\Gamma}_{s,r} = (1, g(s, r))$ . Now, we set  $\varphi : \text{Coker } d \rightarrow \text{Aut } \overline{B}$  by

$$\varphi_s(\overline{b}) = \overline{\theta}_{x_s} \overline{b} (= \overline{\theta_{x_s(b)}}). \quad (8)$$

Since  $x_r x_s = \overline{d}(\tilde{\Gamma}_{r,s}) x_{rs}$ , the functions  $\varphi, g$  satisfy the rule

$$\varphi_s \varphi_r = \mu_{g(s,r)} \varphi_{sr}.$$

Since  $\delta g = 0$ , according to Lemma 8.1 [9] one can defines a crossed product  $E_g = [\overline{B}, g, \varphi, \text{Coker } d]$ . Namely,  $E_g = \overline{B} \times \text{Coker } d$  and the operation on  $E_g$  is

$$(\overline{b}, s) + (\overline{c}, r) = (\overline{b} + \varphi_s(\overline{c}) + g(s, r), sr). \quad (9)$$

In this group  $(0, 1)$  is the zero, while the negative of the element  $(\overline{b}, s)$  is  $(\overline{b}, s^{-1})$ , where  $\varphi_s(\overline{b}) = -\overline{b} - g(s, s^{-1})$ . One obtains an exact sequence

$$\mathcal{E}_g : 0 \rightarrow A \xrightarrow{j_g} E_g \xrightarrow{p_g} D \rightarrow 1,$$

where  $j_g(\zeta(c)) = (\overline{c}, 1)$ ,  $p_g(\overline{b}, s) = db.x_s$ . Indeed,

$$p_g j_g(\zeta(c)) = p_g(\overline{c}, 1) = dc.x_s = 1,$$

and for  $(\overline{b}, s) \in \text{Ker}(p_g)$ , then  $p_g(\overline{b}, s) = db.x_s = 1$ . By the uniqueness of the representation in  $D$ , we have  $db = 1$  and  $x_s = 1$ , it follows that  $b \in \text{Ker } d$  and  $s = 1$ , or  $(\overline{b}, s) \in \text{Im}(j_g)$ .

We prove that  $j_g(A) \subset Z(E_g)$ . For  $b, c \in B$ , one has

$$\mu_{(\overline{b}, s)}(\overline{c}, 1) = (\mu_{\overline{b}} \varphi_s(\overline{c}), 1) \quad (10)$$



If  $c \in \text{Ker } d$ , then by (6),  $\varphi_s(\bar{c}) = \bar{c}$ . Hence,

$$\mu_{(\bar{b},s)}(\bar{c}, 1) = (\mu_{\bar{b}}(\bar{c}), 1) = (\overline{b + c - b}, 1) = (\bar{c}, 1).$$

Since  $j_g(A) \subset Z(E_g)$  and  $p_g$  is a surjection,  $E_g \xrightarrow{p_g} D$  is a crossed module in which the homomorphism  $\theta' : D \rightarrow \text{Aut } E_g$  is the conjugation. To define the morphism  $(\beta, id_D)$  of crossed modules, one set

$$\beta : B \rightarrow E_g, \quad \beta(b) = (\bar{b}, 1).$$

This correspondence is a homomorphism thanks to the relation (9). Clearly,  $p_g \circ \beta = d$ . Moreover, for all  $c \in B$  and  $x = db.x_s \in D$ , we have

$$\beta(\theta_x(c)) = \beta(\theta_{db}(\theta_{x_s}(c))) = (\overline{\mu_b \theta_{x_s}(c)}, 1) \stackrel{(8)}{=} (\mu_{\bar{b}} \varphi_s(\bar{c}), 1).$$

Since  $\theta'_x = \mu_{(\bar{b},s)}$ ,

$$\theta'_x \beta(c) = \mu_{(\bar{b},s)}(\bar{c}, 1) \stackrel{(10)}{=} (\mu_{\bar{b}} \varphi_s(\bar{c}), 1)$$

Thus, the relation  $H_2$  holds, and  $\mathcal{E}_g$  is a  $\zeta$ -extension of co-type  $B \xrightarrow{d} D$ .  $\square$

We state one of the paper's main results.

**Theorem 6.** *Let  $\zeta : \text{Ker } d \rightarrow A$  be the abstract  $\zeta$ -kernel of the crossed module  $B \xrightarrow{d} D$ . Then, the vanishing of the obstruction  $\text{Obs}(\zeta)$  in  $H^3(\text{Coker } d, A)$  is necessary and sufficient for there to exist a  $\zeta$ -extension of co-type  $B \xrightarrow{d} D$ .*

*Proof. Necessary condition.* Let  $\mathcal{E}$  be a  $\zeta$ -extension of co-type  $B \rightarrow D$  satisfying the diagram (4). Then, the reduced categorical group of the categorical group  $\mathbb{G}'$  associated to the crossed module  $E \xrightarrow{p} D$  is  $\text{Red}(1, A, 0)$ . By Proposition 3, the pair  $(\beta, id_D)$  determines a monoidal functor  $(F, \bar{F}) : \mathbb{G} \rightarrow \mathbb{G}'$ . By Proposition 1,  $(F, \bar{F})$  induces a monoidal functor of type  $(0, \zeta)$  from  $\text{Red}(\text{Coker } d, \text{Ker } d, k)$  to  $\text{Red}(1, A, 0)$ . Also by Proposition 1, the obstruction  $[\zeta_* k]$  of the pair  $(0, \zeta)$  vanishes in  $H^3(\text{Coker } d, A)$ .

*Sufficient condition.* It follows directly from Lemma 4 and Lemma 5.  $\square$

## 5 Classification theorem

**Definition.** Two  $\zeta$ -extensions of co-type  $(B, D, d, \theta)$ ,

$$\begin{aligned} 0 \rightarrow A \xrightarrow{j} E \xrightarrow{p} D \rightarrow 1, \quad B \xrightarrow{\beta} E \\ 0 \rightarrow A \xrightarrow{j'} E' \xrightarrow{p'} D \rightarrow 1, \quad B \xrightarrow{\beta'} E' \end{aligned}$$

are *equivalent* if there is an isomorphism  $\omega : E \rightarrow E'$  such that  $\omega j = j'$ ,  $p' \omega = p$  and  $\omega \beta = \beta'$ .

We denote by

$$\text{Ext}_{B \rightarrow D}(D, A, \zeta)$$

the set of all equivalence classes of  $\zeta$ -extensions of co-type  $B \rightarrow D$  inducing  $\zeta$ . We describe this set by means of the set

$$\text{Hom}_{(0, \zeta)}[\text{Red}(D, A, 0), \mathbb{G}/\text{Ker } \zeta]$$

of homotopy classes of monoidal functors of type  $(0, \zeta)$  from  $\text{Red}(D, A, 0)$  to  $\mathbb{G}/\text{Ker } \zeta$ . First, let  $q : B \rightarrow \bar{B} = B/\text{Ker } \zeta$ , and  $\sigma : D \rightarrow \text{Coker } d$  be the natural projections, one states the following lemma.

**Lemma 7.** *If  $\zeta$  is surjective, then the commutative diagram (4) induces a short exact sequence*

$$0 \rightarrow \bar{B} \xrightarrow{\varepsilon} E \xrightarrow{\sigma p} \text{Coker } d \rightarrow 1, \quad (11)$$

where  $\varepsilon(b + \text{Ker } \zeta) = \beta(b)$ .

*Proof.* Obviously,  $\sigma p$  is surjective. It is easy to see that  $\text{Ker } \beta = \text{Ker } \zeta$ , so  $\varepsilon$  is injective. The diagram (4) implies  $\sigma p \varepsilon(\bar{b}) = \sigma p \beta(b) = \sigma d(b) = 1$ , this means that the above sequence is semi-exact. For  $e \in \text{Ker}(\sigma p)$ ,  $p(e) \in \text{Ker } \sigma = \text{Im } d$ , and hence  $p(e) = d(b) = p\beta(b) = p\varepsilon(\bar{b})$ . Then,  $e = \varepsilon(\bar{b}) + ja$ . Since  $ja = j\zeta(c) = \beta(c) = \varepsilon(\bar{c})$ ,  $e = \varepsilon(\bar{b} + \bar{c}) \in \text{Im } \varepsilon$ . Thus, the sequence (11) is exact.  $\square$

**Lemma 8.** *Each  $\zeta$ -extension of co-type  $B \rightarrow D$  is equivalent to a crossed product extension which is constructed from a monoidal functor of type  $(0, \zeta)$ ,  $(\Gamma, \tilde{\Gamma}) : \text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$ .*

*Proof.* Let  $E$  be a  $\zeta$ -extension of co-type  $B \xrightarrow{d} D$ . By the proof of Theorem 6, there is a monoidal functor  $(\Gamma, \tilde{\Gamma}) : \text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$ . By Lemma 5, the crossed product  $E_g$ , where  $g$  is the function associated with  $\tilde{\Gamma}$ , is a  $\zeta$ -extension of co-type  $B \xrightarrow{d} D$  in which

$$\beta_g : B \rightarrow E_g, \quad b \mapsto (\bar{b}, 1).$$

Thanks to the exact sequence (11) in Lemma 7, each element of  $E$  can be represented uniquely as  $\varepsilon\bar{b} + e_s$ , where  $\{e_s, s \in \text{Coker } d\}$  is a set of representatives of  $\text{Coker } d$  in  $E$ . It is easy to check that the correspondence

$$\omega : E \rightarrow E_g, \quad \varepsilon\bar{b} + e_s \mapsto (\bar{b}, s)$$

is a group isomorphism. Moreover,  $\omega$  makes two extensions  $\mathcal{E}$  and  $\mathcal{E}_g$  equivalent.  $\square$

**Theorem 9** (Schreier theory for extensions of co-type of a crossed module).  
*If  $\zeta$ -extensions of co-type  $B \xrightarrow{d} D$  exist, then there is a bijection*

$$\Omega : \text{Ext}_{B \rightarrow D}(D, A, \zeta) \rightarrow \text{Hom}_{(0, \zeta)}[\text{Red}(D, A, 0), \mathbb{G}/\text{Ker } \zeta].$$

*Proof.* The correspondence  $E \mapsto (\Gamma, \tilde{\Gamma})$  in Lemma 8 defines a correspondence  $[E] \mapsto [(\Gamma, \tilde{\Gamma})]$ . The fact that  $\Omega$  is injective implies by following steps.

*Step 1: If monoidal functors  $(\Gamma, \tilde{\Gamma})$  and  $(\Gamma', \tilde{\Gamma}')$  are homotopic, then two extensions  $\mathcal{E}_g$  and  $\mathcal{E}_{g'}$  are equivalent.*

Let  $\Gamma, \Gamma' : \text{Red}(D, A, 0) \rightarrow \mathbb{G}/\text{Ker } \zeta$  be two monoidal functors and  $\alpha : \Gamma \rightarrow \Gamma'$  be a homotopic. Then, the following diagram commutes

$$\begin{array}{ccc} \Gamma s \Gamma r & \xrightarrow{\tilde{\Gamma}} & \Gamma s r \\ \downarrow \alpha_s \otimes \alpha_r & & \downarrow \alpha_{sr} \\ \Gamma' s \Gamma' r & \xrightarrow{\tilde{\Gamma}'} & \Gamma' s r. \end{array}$$

Since the morphisms  $\alpha_s$  are of forms  $(1, a_s)$ , it follows from the above diagram that

$$g(s, r) - g'(s, r) = a_s + a_r - a_{sr} = (\delta a)(s, r). \quad (12)$$

Since  $\zeta$  is surjective,  $a_s = \zeta(z_s)$ , where  $z : \text{Coker } d \rightarrow \text{Ker } d$  is a normalized function.

Then, by (12),  $\alpha$  determines a map  $\omega : E_g \rightarrow E_{g'}$  by

$$(\bar{b}, s) \mapsto [\overline{b + z_s}, s]. \quad (13)$$

By the relation (6) and by the definition of operations in  $E_g, E_{g'}$ , the map  $\omega$  is a group homomorphism. Further, it makes two extensions  $\mathcal{E}$  and  $\mathcal{E}_g$  equivalent.

*Step 2: If two extensions  $\mathcal{E}_g$  and  $\mathcal{E}_{g'}$  are equivalent, then  $(\Gamma, \tilde{\Gamma})$  and  $(\Gamma', \tilde{\Gamma}')$  are homotopic.*

Let  $\mathcal{E}_g$  and  $\mathcal{E}_{g'}$  be equivalent via the isomorphism  $\omega : E_g \rightarrow E_{g'}$ . From  $p_g = p_{g'}\omega : E_g \rightarrow E_{g'} \rightarrow D$ , it follows that  $\omega$  is of the form (13), where  $z : \text{Coker } d \rightarrow \text{Ker } d$  is a normalized function. Since  $\omega$  is a homomorphism,  $\alpha = \zeta_* k$  is a homotopy between  $\Gamma$  and  $\Gamma'$ .

It follows from Lemma 5 that  $\Omega$  is surjective.  $\square$

It follows from Proposition 1 and Theorem 9 that

**Corollary 10.** *If  $\zeta$ -extensions of co-type  $B \xrightarrow{d} D$  exist, then there is a bijection*

$$\text{Ext}_{B \rightarrow D}(D, A, \zeta) \leftrightarrow H^2(\text{Coker } d, A).$$

## 6 Prolongations of a group extension

In this section we show an application of  $\zeta$ -extensions of co-type of a crossed module in order to obtain the results on prolongations of a group extension in the sense of [13]. Given a commutative diagram of group homomorphisms

$$\begin{array}{ccccccc} \mathcal{B} : & 0 & \longrightarrow & \text{Ker } \pi & \xrightarrow{i} & B & \xrightarrow{\pi} \Pi \longrightarrow 1 \\ & & & \downarrow \zeta & & \downarrow \beta & \downarrow \eta \\ \mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{j} & E & \xrightarrow{p} D \longrightarrow 1 \end{array} \quad (14)$$

where the rows are exact,  $\text{Ker } \pi \subset ZB$ ,  $\eta$  is a normal monomorphism (in the sense that  $\eta\Pi$  is a normal subgroup of  $D$ ) and  $\zeta$  is an epimorphism. Then,  $\mathcal{E}$  is said to be a  $(\zeta, \eta)$ -prolongation of  $\mathcal{B}$ .

For the quotient group  $\overline{B} = B / \text{Ker } \zeta$ , the homomorphisms  $i, \eta\pi, \zeta, \beta$  in the commutative diagram (14) induce the homomorphisms  $\iota, d, \overline{\zeta}, \overline{\beta}$ , respectively, such that the following diagram commutes

$$\begin{array}{ccccccc} & & \text{Ker } d & \xrightarrow{\iota} & \overline{B} & \xrightarrow{d} & D \\ & & \downarrow \overline{\zeta} & & \downarrow \overline{\beta} & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{j} & E & \xrightarrow{p} & D \longrightarrow 1, \end{array} \quad (15)$$

Besides, according to Theorem 2 [13],  $\mathcal{E}$  induces a homomorphism  $\theta : D \rightarrow \text{Aut } \overline{B}$  such that the quadruple  $(\overline{B}, D, d, \theta)$  is a crossed module.

**Theorem 11.**  $\mathcal{E}$  is a  $\overline{\zeta}$ -extension of co-type  $(\overline{B}, D, d, \theta)$ .

*Proof.* In the diagram (15), since the bottom row is exact and  $jA \subset ZE$  (Theorem 10 [13]), the epimorphism  $p : E \rightarrow D$  together with the conjugation in  $E$  is a crossed module. It is easy to see that the pair  $(\overline{\beta}, id_D)$  is a homomorphism of crossed modules, so  $\mathcal{E}$  is a  $\overline{\zeta}$ -extension of co-type  $(\overline{B}, D, d, \theta)$ .  $\square$

- *The problem of prolongations of a group extension.*

Given a diagram of group homomorphisms

$$\mathcal{E} : \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \pi & \xrightarrow{i} & B & \xrightarrow{\pi} & \Pi \longrightarrow 1 \\ & & \downarrow \zeta & & & & \downarrow \eta \\ & & A & & & & D \end{array}$$

where the row is exact,  $i$  is an inclusion map,  $\text{Ker } \pi \subset ZB$ ,  $\eta$  is a normal monomorphism,  $\zeta$  is surjective, and a group homomorphism  $\theta : D \rightarrow \text{Aut}(\overline{B})$  such that the quadruple  $(\overline{B}, D, d, \theta)$  is a crossed module (where the notations  $\overline{B}, d$  are defined as above). These data are denoted by the triple  $(\zeta, \eta, \theta)$ ,

called a *pre-prolongation* of  $\mathcal{E}$ . A  $(\zeta, \eta)$ -prolongation of  $\mathcal{E}$  inducing  $\theta$  is also called a *covering* of the pre-prolongation  $(\zeta, \eta, \theta)$ .

The “prolongation problem” is that of finding whether there is any covering of the pre-prolongation  $(\zeta, \eta, \theta)$  of  $\mathcal{E}$  and, if so, how many.

According to [13], each pre-prolongation  $(\zeta, \eta, \theta)$  of  $\mathcal{E}$  induces an obstruction  $k$ . This obstruction is just the obstruction of an abstract  $\zeta$ -kernel of the crossed module  $\overline{B} \xrightarrow{d} D$ . Thus, from the results on crossed modules in previous sections, one obtains the solution of the problem of prolongations of a group extension (Theorem 8 and Theorem 15 in [13]).

**Theorem 12.** *Let  $(\zeta, \eta, \theta)$  be a pre-prolongation.*

i) *The vanishing of the obstruction  $[\overline{\zeta}_* k]$  in  $H^3(\text{Coker } d, A)$  is necessary and sufficient for there to exist a covering of  $(\zeta, \eta, \theta)$ .*

ii) *If  $[\overline{\zeta}_* k]$  vanishes, there is a bijection*

$$\text{Ext}_{(\zeta, \eta)}(D, A) \leftrightarrow H^2(\text{Coker } d, A),$$

where  $\text{Ext}_{(\zeta, \eta)}(D, A)$  is the set of equivalence classes of  $(\zeta, \eta)$ -prolongations of the extension  $\mathcal{B}$  inducing  $\theta$ .

*Proof.* i) According to Theorem 6, the vanishing of  $[\overline{\zeta}_* k]$  in  $H^3(\text{Coker } d, A)$  is necessary and sufficient for there to exist a  $\overline{\zeta}$ -extension  $\mathcal{E}$  of co-type  $\overline{B} \xrightarrow{d} D$ . Thanks to the following diagram, this is equivalent to the fact that  $\mathcal{E}$  is a covering of the pre-prolongation  $(\zeta, \eta, \theta)$ ,

$$\begin{array}{ccccccc} \mathcal{B} : & 0 & \longrightarrow & \text{Ker } \pi & \xrightarrow{i} & B & \xrightarrow{\pi} \Pi \longrightarrow 1 \\ & & & \downarrow \zeta_0 & & \downarrow p_0 & \downarrow \eta \\ & & & \text{Ker } d & \xrightarrow{\iota} & \overline{B} & \xrightarrow{d} D \\ & & & \downarrow \overline{\zeta} & & \downarrow \overline{\beta} & \parallel \\ \mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{j} & E & \xrightarrow{p} D \longrightarrow 1. \end{array}$$

ii) It is clear that two coverings of the pre-prolongation  $(\zeta, \eta, \theta)$  are equivalent if and only if they are two equivalent  $\overline{\zeta}$ -extensions of co-type  $\overline{B} \xrightarrow{d} D$ , that is, there is a bijection

$$\text{Ext}_{(\zeta, \eta)}(D, A) \leftrightarrow \text{Ext}_{\overline{B} \rightarrow D}(D, A, \overline{\zeta}).$$

Now, by Corollary 10, we have the bijection

$$\text{Ext}_{(\zeta, \eta)}(D, A) \leftrightarrow H^2(\text{Coker } d, A). \quad \square$$

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